LINEAR TRANSFORMATIONS WITH CABRI II VIA MAPLEV.
A FRIENDLY REPLY TO ANNA SIERSPINSKA.

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I. Introduction. -

What is a linear transformation?

The usual definition found in any textbook of Linear Algebra is:

A transformation T defined on a vector space V such that
(i) T(cv) = cT(v) for any scalar c and vector v
(ii) T(v+w) = T(v) + T(w) for any pair v,w of vectors.

Which mode of interpretation chose our students to grasp this definition? First, the word *'transformation'* has a metaphorical meaning; it indicates a change and even a movement on the space.
Then how are understood the two conditions **(i)&(ii)?** geometrically? arithmetically? or algebraically?
These three modes of interpretation were observed and analyzed by several authors
(cf. A. Defence, T. Dreyfus, J. Hillel, A. Sierpinska, & S. Khatcherian).

II. Experience with Cabri II of Anna Sierpinska's Team. –

Ref. Cabri Based Linear Algebra: Transformations by Tommy Dreyfus, Joel Hillel and Anna Sierpinska

The experience was run in an anglophone cégep of Montréal with students who already attended a "baby linear algebra course".
Students in anglophone cegeps, using mostly Anton's book, are learning Linear Algebra in the framework of the arithmetic language of $\mathbf{R}^n$ and the algebra of matrices**.** Unfortunately only later, towards the end of the course, the geometry appears through the presentation of examples of linear transformations: projections, mirror symmetries, rotations, homotheties, etc…Some ambiguity then gets installed in the students' minds; how to identify 2x2-matrices with linear transformations on $\mathbf{R}^2$? They work with vector spaces equipped with bases without being aware of it. The experience run by the team of Anna Sierpinska uses Cabri II as the decoder tool. The students were shown that a linear transformation T can be defined by a set of 2 pairs of vectors (u1, u2) and (v1, v2). The macro-construction was not explained. Students were expected first to understand the relation between the two pairs of vectors inherent to T by moving the given

vector w to u1 and u2 and find out the property: T(u1)=v1 and T(u2) = v2. Then they should be able to use the conditions (i) & (ii) to extend the construction to T(w) for any vector w by using: T(w) = c1 v1 + c2 v2 if w = c1 u1 + c2 u2.
Even though Cabri allowed them to move all vectors (except T(w)), the students expressed their confusions, when talking during the task, between the transformation T and the image T(w); they were unable to find out positions for v1 and v2 when asked to project any vector w onto a given line. Was Cabri not a good environment for linear transformations or only had the scenario missed a point?

III. Experience with Maple V

I will describe now my own pedagogical scenario for teaching the concept of linear transformations that I have been using since 10 years. My students being first exposed heavily to the algebra of vectors and matrices, I use matrices as prototypes of linear transformations. The two conditions (i) & (ii) are easily accepted as coming from the properties of the algebra of matrices:

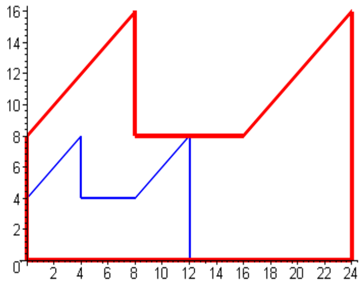
(i') A . (cu) = c (A.u)
(ii') A . (u+v) = A . u + A . v

During a 2 hour workshop, ten years ago with grid papers and pencils, now in a computer-lab with CASs as Derive and Maple, I gave to students a set of 2x2-matrices together with a 2xn-matrix representing a closed polygon with n vertices, one of which is the origin (0,0); this polygon has few right angles and pairs of parallel sides.
Students are asked to plot first the initial polygon, evaluate its area, look at its orientation when following the order in the matrix; then for each given matrix they repeat the same task: plot, area, orientation of the new polygon equal to the image under the transformation studied; they are to collect all observations into a big tableau with initial entries equal to the given 2x2-matrices; those observations are about the preservation of parallelism and the origin for each linear transformation, preservation of right angles only for symmetries, homotheties and rotations, change in areas and orientation depending on the determinant of the matrix of the transformation. Finally they are asked to write their own matrix with determinant = 0 and find out what happens in such a case to the image of a closed polygon. Students can see easily that the 2 column vectors of their matrix span the line of projection onto which the image has collapsed .
After such a workshop, during regular class time we can discuss with more comprehension and depth on the geometric role of the conditions: (i) & (ii):

- The image of any line through the origin is again such a line.
- The image of any pair of parallel lines is again a pair of parallel lines.
- The image of any closed polygon is again a closed polygon or a closed interval in the case of a singular matrix.

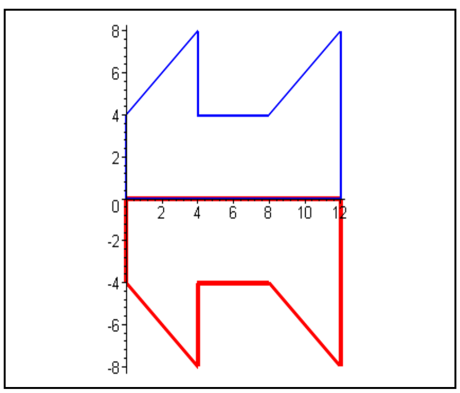
Finally the role of the matrix as a code for a geometric transformation is clarified by exhibiting the initial basis (u1, u2) (standard basis is commonly used here) and the image pair (v1, v2) = (T(u1),T(v1)) that is also a basis if the transformation is not singular, i.e. is invertible or equivalently of determinant not equal to 0.

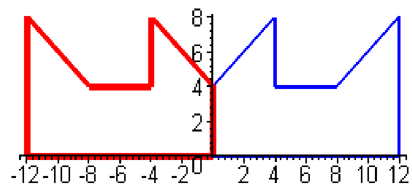
Using a CAS as Derive or Maple helped me to diminish the ambiguity in the student's minds between the arithmetic and geometric modes of representation of a linear transformation.
Here are few of the graphics obtained with Maple.

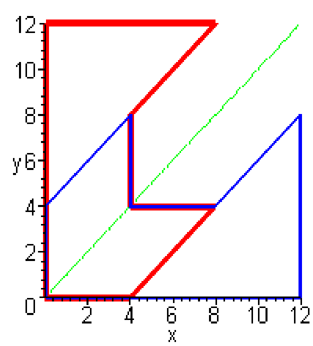


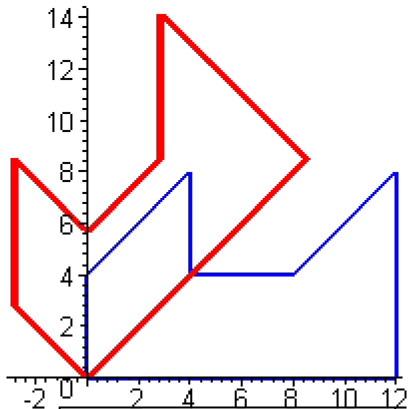
Homothetie of ratio 2

Here are three mirror symmetries: x- axis, y-axis and then the line y=x.

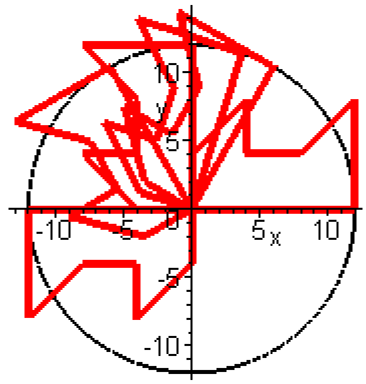




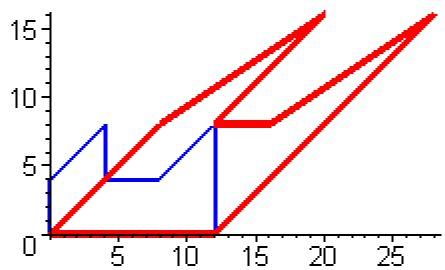




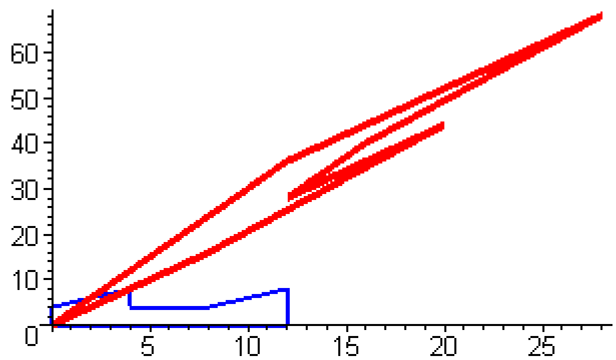
Rotation by 45 degrees



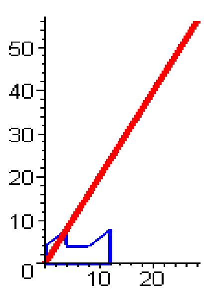
Few rotations by an angle at most equal to 180 degrees



Shear Transformation
along the x-axis encoded by the matrix G equal to [1 2]
[0 2]



A general linear transformation encoded by the matrix H equal to
[1 2]
[3 4]



With a matrix of determinant equal to 0, the linear transformation collapses the polygon onto a line spanned by a vector proportional to both column vectors of the matrix

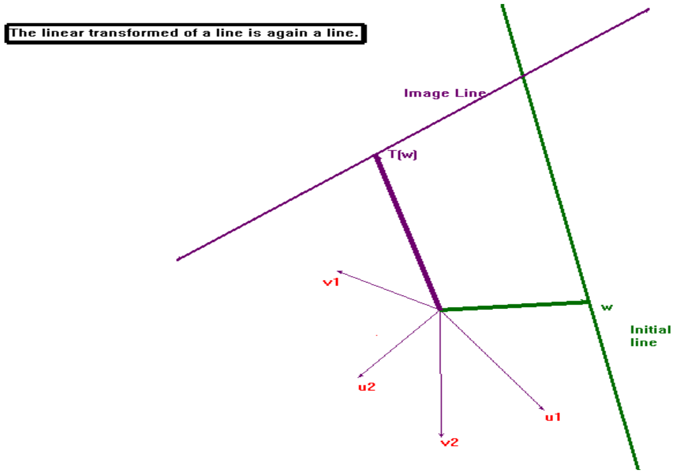
[1 2]
J =
[2, 4]

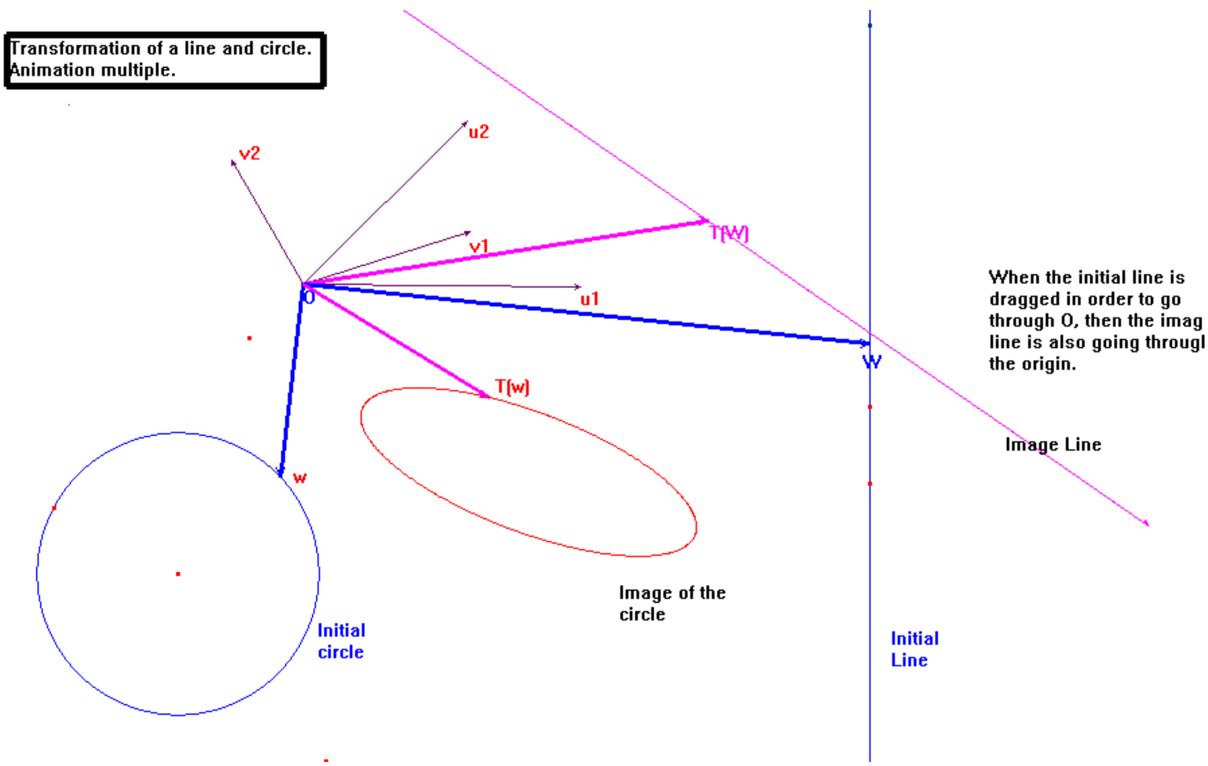
IV. Project with Cabri II for the Fall 2001.

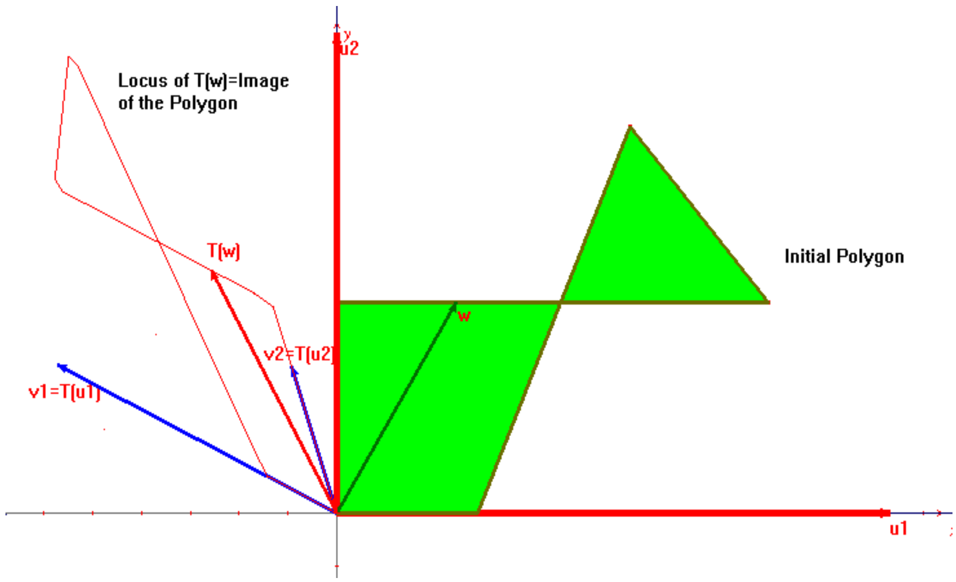
Next semester (Fall 2001) I will be able to use CABRI II at Dawson College, during my Linear Algebra classes. I will adapt to the Cabri environment, the pedagogical scenario described in the previous paragraph.
For the computer-lab the macro-construction of a linear transformation depending on the origin O, an initial basis (u1, u2), the image pair (v1, v2) will be given for use with its help but not explained to students; only later during regular class time the macro construction will be analyzed and the importance of the conditions: (i) & (ii) will be stressed out.
I might give again to students the same set, as during the Maple exercise, of 2x2-matrices decoding well-known linear transformations. Using the standard set of axes and associated grid of Cabri, students may draw their own closed polygon having O as one vertex, with parallel sides and few right angles. Then they point one vector w onto this polygon. To be able to use the macro of linear transformation, they will choose the standard basis for (u1, u2) and then draw two vectors v1 and v2 originating from O. They should be warned that the pair (v1, v2) represents the image pair (T (u1), T(u2)) that happens to be the two column vectors of the 2x2-matrix representing T. Now the superb locus function of Cabri is able to trace the whole image of the polygon under the transformation T. How to change the transformation T? With Cabri it is a very easy task, as the student just needs to move the 2 vectors v1 and v2. For each matrix given during the Maple task, here we just need to put the vectors v1 and v2 into the positions of the column vectors of the matrix. Then as in a dream, the previous locus changes simultaneously to the new image polygon. More over we may ask the students to use the animation function for two reasons:

- To observe the simultaneous moves of w onto the initial polygon and T(w) on the image. Parallelism, right angles and orientation could be analyzed during this animation. Indeed when the transformation is preserving the orientation, as for rotations, during the animation we can observe the movements of w and T(w) in the same direction; This could not be done with Maple.
- Given a parametric family of linear transformations, as rotations in the last Maple experiment, we can animate a move of the pair (v1, v2) together with the family of images of the polygon.

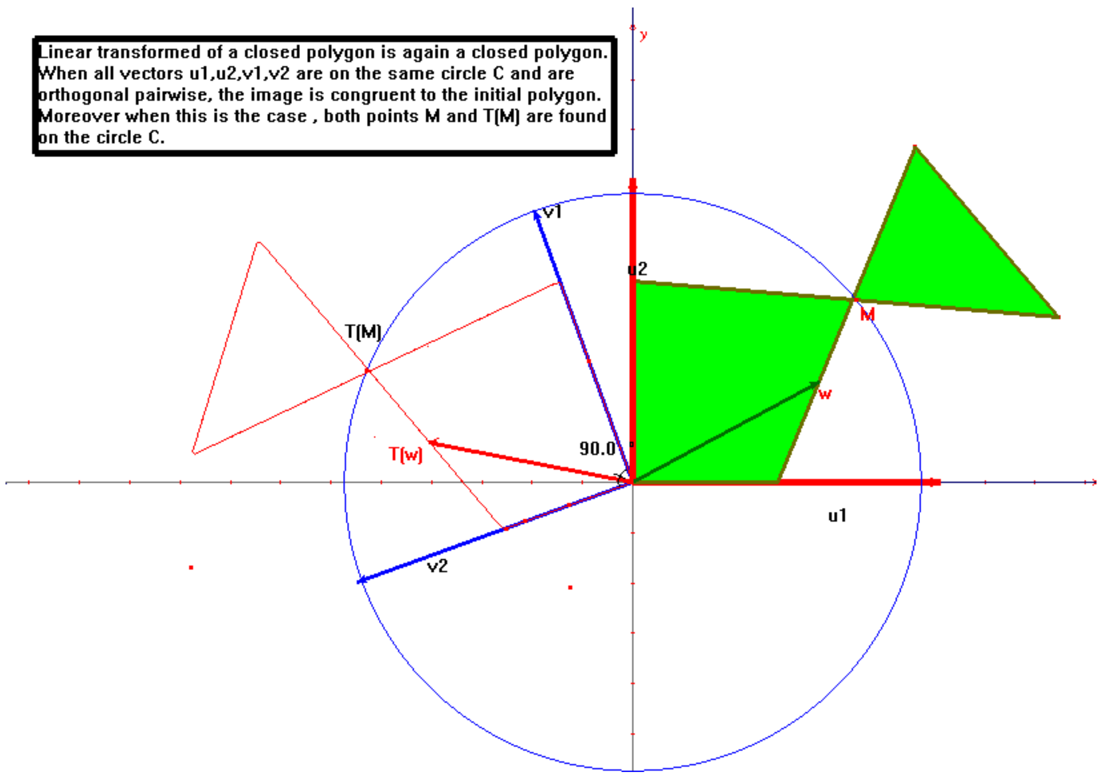
It seems that with this new scenario with Cabri, we should gain in clarity for the geometric relationships to observe. Finally I should address back to the problem of confusion observed by the team of Anna Sierpinska between T and T(w). I expect that the students will confuse less as T here is a procedure hidden behind a macro and T(w) belongs to the locus and was constructed. Also the task to find out in which positions to put v1 and v2 in order to project the polygon onto a given line should be judged easy by students who know well the significance of the vectors v1 and v2 as images of u1 and u2 under T. Because of the plasticity and transparency of Cabri, I claim that the treatment Maple-Cabri should cure the students from their tendencies to confuse procedure with its result.


The linear transformed of a line is again a line.
Image Line
T(w)
v1
w
Initial
line
u2
u1
v2

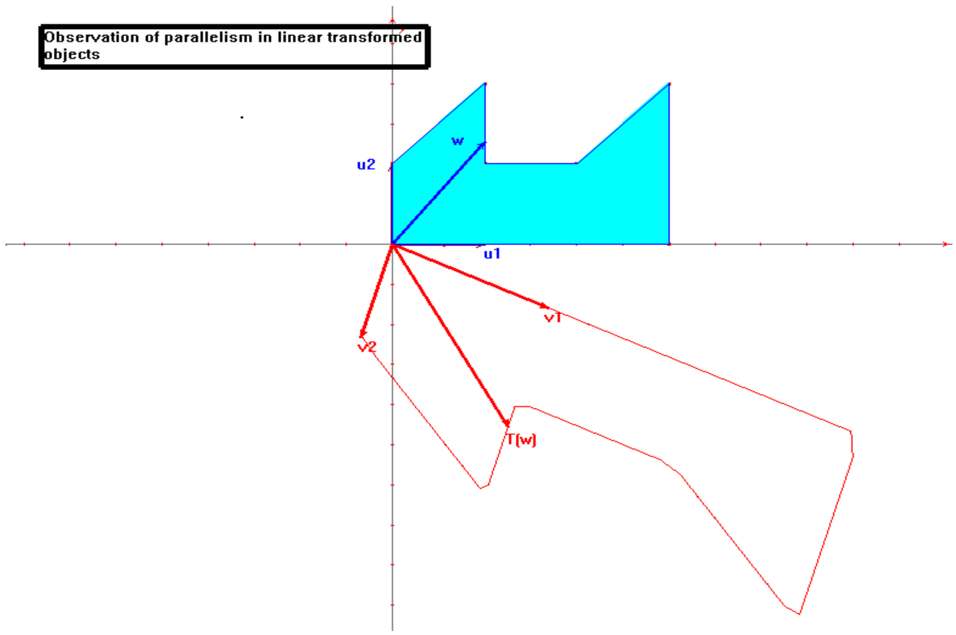

Transformation of a line and circle.
Animation multiple.
u2
v2
v1
T(W)
O
u1
W
T(w)
w
When the initial line is
dragged in order to go
through O, then the imag
line is also going througl
the origin.
Image Line
Initial
circle
Image of the
circle
Initial
Line

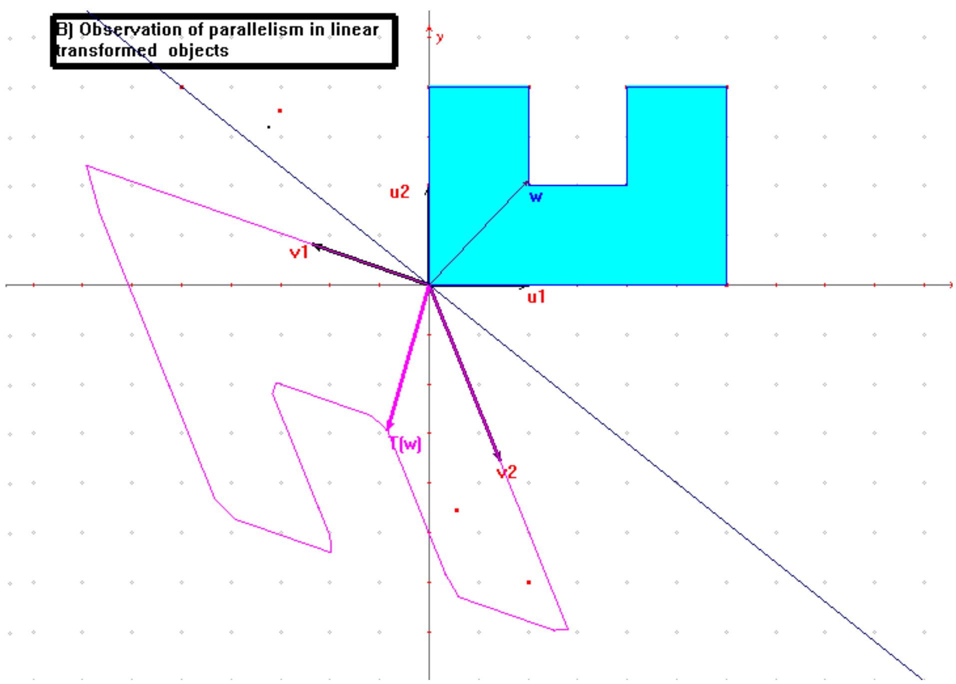


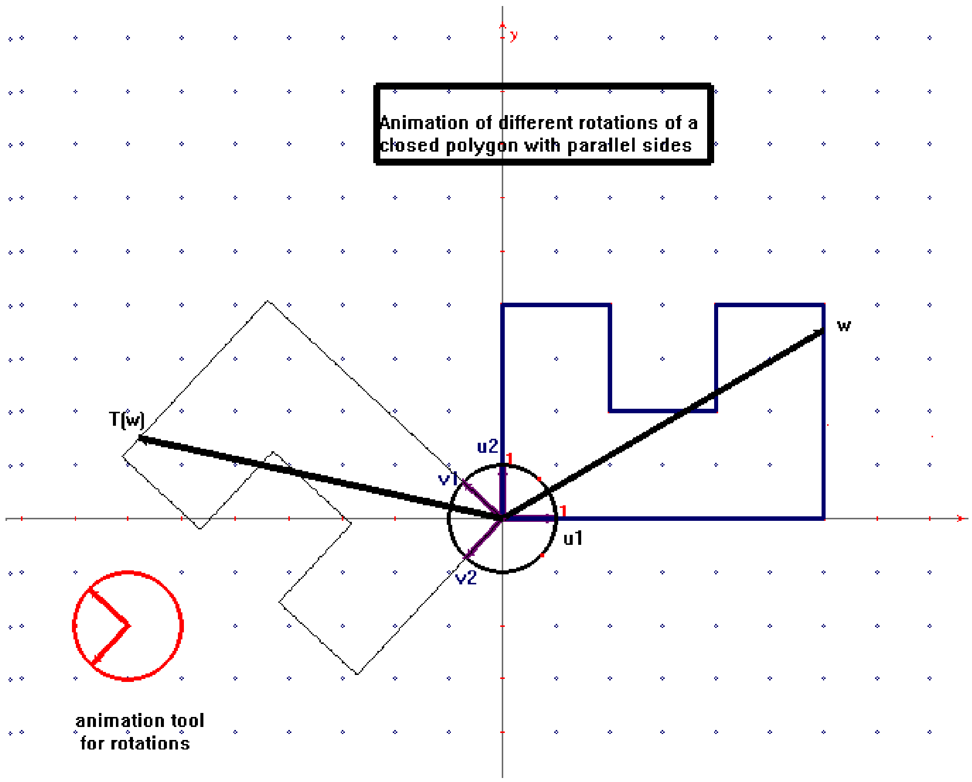
y
u2
Locus of T(w)=Image
of the Polygon
Initial Polygon
T(w)
w
v2=T(u2)
v1=T(u1)
u1

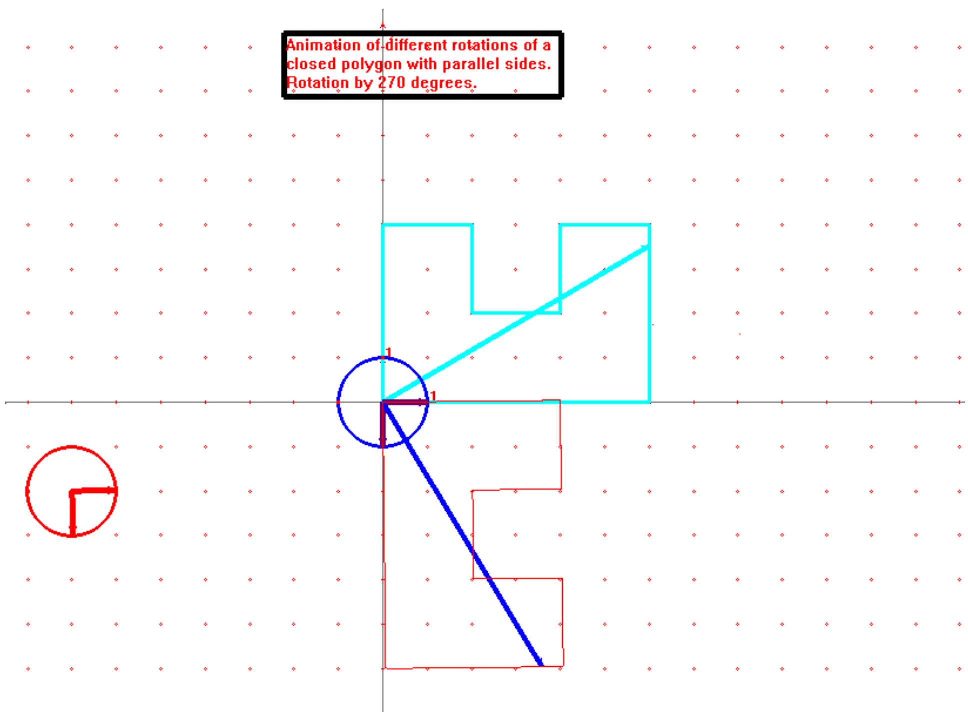


Linear transformed of a closed polygon is again a closed polygon. When all vectors u1,u2,v1,v2 are on the same circle C and are orthogonal pairwise, the image is congruent to the initial polygon. Moreover when this is the case , both points M and T(M) are found on the circle C.
y
v1
u2
M
T(M)
w
T(w)
90.0
u1
v2


Observation of parallelism in linear transformed objects
w
u2
u1
v1
v2
T(w)


B) Observation of parallelism in linear transformed objects
y
u2
w
v1
u1
T(w)
v2


y
Animation of different rotations of a closed polygon with parallel sides
w
T(w)
u2
v1
1
1
u1
v2
animation tool
for rotations


Animation of different rotations of a closed polygon with parallel sides. Rotation by 270 degrees.
1
1

Bibliography:

- **Cowen Carl C.** (1997), *On the Centrality of Linear Algebra in the Curriculum,* MAA on line http://www. maa.org/features/cowen**,** on receiving the Deborah and FranklinTepper Haimo Award for Distinguished College or University Teaching of Mathematics, San Diego California, January 1997.

- **Dreyfus,T., J. Hillel & A. Sierpinska,** (1997), *Coordinate-free geometry as an entry to linear algebra.* In M. Hejny & J. Novotna(eds), Proceedings of the European Research Conference on Mathematical Education, pp.116-119. Podebrady/Prague, Czech Republic.

- **Dreyfus,T., J. Hillel & A. Sierpinska,** (1997), *A propos de trois modes de raisonnement en algebre lineaire en question,* Panorama de la Recherche en Didactique sur ce theme (pp.249-268), Grenoble, France, La Pensée sauvage.

- **CMS 99 Winter Meeting, Montreal,** Education Session on Teaching of Linear Algebra, December 11-13, 1999.

    - **Auer, J.,** Brock University, St Catharines, *Ten years of teaching Linear Algebra with Maple V.*
    - **Byers, B.,** Concordia University, Montreal, *Working with ambiguity in linear algebra.*
    - **Klasa Jacqueline,** animatrice du **Forum on teaching Linear Algebra,** two invited speakers: **Vincent Papillon,** College Brebeuf, Montreal, & **John Labute,** McGill University, Montreal**.** Rapport bilingue.
    - **Lay David**, University of Maryland, College Park, USA. Plenary Conference: *Recent Advances in Teaching Linear Algebra.*
    - **Oktac Asuma,** Concordia University, Montreal, *Linear Algebra: Is it possible at a distance?*
    - **Norman D.,** Queen's University, Kingston, *Teaching linear algebra independence via unique representation.*
    - **Sierpinska Anna,** Concordia University, Montreal, *Practical, theoretical, synthetic and analytic modes of thinking in linear algebra.*

- **ACDCA 2000- Austrian Center for Didactics of Computer Algebra, Portoroz, Slovenia, July 2-5, 2000.**

    - **Klasa J. & S.,** Dawson College & Concordia University, Montreal, *Linear Algebra with Maple.*