The stereographic projection: Modelation and exploration with Cabri-Geometry

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Abstract
In this article we show how Cabri-Geometry is a powerful tool for the visualization of diverse notions around the stereographic projection. It becomes patent how, in spite of the use of a representation in perspective cavalier of the projection sphere, the knowledge that are generated keep an epistemic fidelity regarding those that are obtained of the respective algebraic representations. If to this it is added that the group of examples that the traditional curriculum approaches is very limited, the use of this package of Dynamic Geometry as means for the exploration of this transformation is advantageous for the presentation of this topic inside the classroom. We conclude with a proposal for the visualization of a topologic theorem.

Introduction

The stereographic projection is the transformation of the complex plane into the sphere that satisfies the following: with the South Pole of the sphere supported on the origin of the complex plane, the transform associates a point of the sphere with a point of the plane, for a straight line that connect the point and his image with the North Pole. The correspondence is the point of intersection of the straight line with the sphere; to the North Pole of the
sphere (only exception in this association rule) corresponds the point to the infinite in the plane.
The inverse correspondence of points of the sphere toward those of the plane is made considering the straight line that goes by the North Pole and the point given in the sphere, which is equivalent to the rule described previously. Let us make notice that the inclusion of this point to the infinite \(\infty\) like part of the plane allows to extend the algebraic laws so they cover certain operations that involve the symbol \(\infty\) without producing contradictions in the algebra and ordinary calculation (Schwerdtfeger, 1979, pgs. 22-34).
The group of problems that are approached can usually be synthesized in the following way: given a curve, which is their image under this transformation (either the direct or inverse transformation)? Now then, the treated curves don't go beyond segments of right, rays, complete straight line, arches of circles or circumferences. An analysis a priori of the reasons that it happens, would indicate that difficulties exist to extend the class of the curves in study, to mention some, their grade of complexity from the point of view of algebraic, as much for their form as for their symbolic manipulation. Also, from a geometric perspective, the obtaining of a graphic representation is a delicate point. All this will be exhibited in the following sections.

**Construction of a microworld of exploration in Cabri-Geometry**

To expand our perception about the stereographic projection, it is desirable the construction of a microworld that allows us to have images of the previous results and also to intrude in not explored territory.

The microworld consists of two parts, the first one corresponds to the complex plane and the second correspond to the representation in perspective cavalier simultaneous of the sphere of stereographic projection and the complex plane. In Cabri-Geometry it is possible to have both views in the same work area or in independent parts; we will take advantage of that easiness in this work.

The part corresponding to the complex plane works in such way that are associated to a point \(z_1\) of the plane the modulus of its vector radio and the arch inside the unitary circle corresponding to the main argument of the complex \(z_1\). The arch will allow to locate in the projection sphere to the meridian on which we will have the image of the point \(z_1\). The modulus of the vector \(z_1\) is the size, in the corresponding perspective. Finally, to draw the intersection of the meridian with the termination of this vector in perspective, which will be the point image. For a more precise reference about the construction of the sphere in perspective cavalier, can be consulted (Rousselet, 1995).

**Right and transformed circumferences: the traditional teaching.**

To obtain algebraically the function that send the points from the plane to the projection sphere, we consider the following:
1) The equation of the sphere with center in \(C (0, 0, 1)\) and radius 1,
2) The equation of line that goes by the North Pole \(N (0, 0, 2)\), and,
3) The point in the complex plane \(z_1 = x_1 + i y_1\).
This way, the equation of the sphere that has the requested characteristics turns out to be then \( x^2 + y^2 + (z-1)^2 = 1 \). The canonical equation of the straight line that goes by the North Pole and the point \( z_1 \) of the plane is

\[
\frac{x}{\text{Re}(z_1)} = \frac{y}{\text{Im}(z_1)} = -\frac{z-2}{2}
\]

Similarly to the resolution presented in (Schwerdtfeger, 1979), you arrive to that the coordinates of the point in the sphere \( P (x, y, z) \) well-known the point \( z_1 = x_1 + i y_1 \) of the complex plane, they are given by the following relationships:

\[
x = \frac{4x_1}{z_1 \bar{z}_1 + 4}, \quad y = \frac{4y_1}{z_1 \bar{z}_1 + 4}, \quad z = \frac{4z_1 \bar{z}_1}{z_1 \bar{z}_1 + 4}
\]

Inversely, if we want to obtain the point \( z_1 = x_1 + i y_1 \) of the complex plane that corresponds to a point \( P (\xi, \eta, \zeta) \) specific of the sphere, they are the following conditions:

\[
x_1 = \frac{2\xi}{2 - \xi}, \quad y_1 = \frac{2\eta}{2 - \xi}
\]

Known these algebraic transformations, we will calculate toward where the image of a point \( z_1 \) goes when that the point goes to the infinite point \( \infty \) (that is, it moves on a straight line that goes by the origin and has a value of slope the real number \( k \)).

If we call \( g \) to the function that sends points from the complex plane to the projection sphere, under the conditions of the previous paragraph we have that \( z_1 = x_1 + i k x_1, z_1 \) tends to \( \infty \), and then:

\[
\lim_{z_1 \to \infty} g(z_1) = (\lim_{x_1 \to \infty} \frac{4x_1}{(k^2 + 1)x_1^2 + 4}, \lim_{x_1 \to \infty} \frac{4kx_1}{(k^2 + 1)x_1^2 + 4}, \lim_{x_1 \to \infty} \frac{2(k^2 + 1)x_1^2}{(k^2 + 1)x_1^2 + 4}) = N(0,0,2)
\]

It can be demonstrated that this is equal to find relationships among similar triangles in the space. These algebraic methods are, if not the only ones, if those more frequently reported in the traditional books (Eves, 1985; Schwerdtfeger, 1979; Churchill, R., Brown, J. 1985). Inside such a bibliography, the results that they are pointed out as the most important, we find the following ones:

1. The meridians of the sphere correspond to the straight line that go by the south pole and the parallel ones correspond to concentric circumferences whose center is also the south pole. In particular, the equator corresponds to the circumference whose radio is the diameter of the sphere, in this case 2.
In the figure has been drawn the complex plane and a circumference simultaneously in perspective cavalier to study where the point $z_1$ moves freely (to the right). The representation has also some features of the projection sphere (the equator and their reflection in the plane, a fixed meridian and another mobile, an equatorial polar and other radius, the poles). The interaction with the circle free of the left, which moves $z_1$, it allows us to identify the result partly that it has been enunciated previously. A possibility is the overlapping of conical to the geometric places of the representation in perspective cavalier, to corroborate that in any event we have ellipses (the equation command and coordinates are advisable also).

2. The circumferences that go by the North Pole of the sphere correspond to straight lines of the plane.
We can explore the images of the straight line in the projection sphere without necessity of reconstructing the microworld, using the command of redefinition of objects of the environment Cabri-Geometry, sending to the free point $z_I$ from the circumference to a free straight line of the complex plane. In passing with the exploration of this microworld the identification will be completed of curved and pending images of the first result. Also, using the call method of the funicular one, described for another purpose in (Díaz Barriga, 2000), we can send elements to the infinite and to observe the transitions among different cases. As guides of exploration activities we suggest to carry out the following actions:

a) Sent a free point $z_I$ of the complex plane gradually to the infinite and observe the coordinates of their image.

b) Trace a circumference with center OR and that it goes by a fixed point K. Obtain the image of this circumference I under stereographic projection. Then it sent the center OR to the infinite.

c) Trace a circle arch that goes by three points A, B and C. Obtain the image of each one of them. Then it sent each point to the infinite, distinguishing the cases of extreme points of the arch of circle of the case of in the one that the point is intermediate point in the arch.

3. The angle formed by two straight line in the plane is similar to the angle formed by its stereographic projections in the sphere.

The exploration of the transformations that the angles suffer under stereographic projection it can be made under the transformation of triangles in the complex plane and again to use the command of redefinition of objects.

Here we can delimit that we will meet with the fact that the perspective cavalier includes an additional factor to observe this measures. The construction of a spherical trigonometry that allows to settle down distances and angles for points in the sphere will be outside of the aspirations of this work.
4. If the points $z_1'$ and $z_2'$ of the sphere they are reflections one of the other one in the equatorial plane of the sphere, then their images $z_1$ and $z_2$ on the plane are inverse one of the other first floor the image in the plane of the equator.

To conclude the relationship between the points $z_1$ and described $z_2$, it is necessary to consider that the one modulates of conical with which Cabri-Geometry counts it solves the intersections appropriately with the straight line and the segments. Of here it already takes advantage the mobile meridian built for the stereographic projection, a parallel one is traced to the polar diameter that goes by the point $z_1'$. The intersection with the conical one is called $z_2'$, point that is used stops later to obtain its image $z_2$ in the plane; soon after the inversive relationship is corroborated in the complex plane, via proportionality of segments with the calculator and the command inversion.

5. The circumferences that don't pass for the North Pole of the sphere correspond to ellipses of the plane.
Again we can explore the images of an ellipse given in the complex plane without making a new construction for the action of the command of redefinition of objects of the environment Cabri-Geometry. Restricting to the point $z_1$ to move in her, we can verify their image in the projection sphere.

**Some more words: rolling circles.**

The sphere that we have chosen for our stereographic projection only has some very identified details (equator, poles, polar diameter), reason why it shines a transparent point to our curious eyes.

There is a curve that deserves a special mention: the cycloid, the hypocycloid and the epicycloid. The construction of their corresponding images offers us elements to make a deeper reflection from the inherent properties to the stereographic projection. Let us remember the following definitions:

When a radio circle to wheel outwardly (inwardly) without sliding on a fixed circumference of radio $b$, to the trajectory of a point $P$ of the rolling circumference is called epicycloid (hypocycloid).

When a radio circle to wheel without sliding on a straight line, to the trajectory of a point $P$ on the rolling circumference is called cycloid. If we consider a point $M$ between the center of the rolling circumference and the point $P$, collinear with them, to its trajectory is called contracted cycloid (Weisstein, 1999). If the distance between $M$ and the center is bigger
than the radius of the rolling circle, the trajectory is called extended cycloid (Weisstein, 1999).

In the figure two possible curves are illustrated to explore their images I under stereographic projection. The first curve, that gradually can be contracted or extended for the point I, has to the straight line and the cycloid inside the family of curves can be generated grace to the dynamics of Cabri-Geometry. This allows have objects that serve continually from reference to the transform the domain of curved in study. This is also the case of the second whose image is illustrated in the sphere; in this case the circumference can roll interior or exteriorly, generating like it curves of reference a circumference, the family of curved of the domain allows to explore hypocycloids and contracted or extended epicycloids.

A stamen to see a topological theorem: the Archimedes’ spiral.

The discoveries are patent inside our microworld by means of their exploration with different classes of curves. When we wonder for the image of an Archimedes’ spiral in the sphere of projection whose election owes her especially to two properties that the curve has in the plane. The properties are:
1. The branches of the spiral are equally spaced.
2. As the one numbers of turns it grows, the curve moves away from the starting point

In the construction we have used a spiral of type archimedian, where the North is a free point, the circumference has arbitrary radio, the arch RS is measured in positive sense in this circumference. It has the same longitude of the segment that goes of North/ South to z1, being able to this arch to coil several times around this circumference. They are also presented the circumferences of radio one and two, to locate the location of the equator of the sphere and their image better in the complex plane with regard to the origin and the axes of coordinated (coincident with the poles north and south).
What images are they observed in the projection sphere? By means of the dynamics that Cabri-Geometry offers us, we can control the spacing among the branches of the spiral and to go filling the plane. If we diminish the radius of the circumference centered in North, we reduce the spacing among the branches of the spiral in the plane. If we encourage the point S of the arch RS, we give turns to the spiral and their branches sweep the plane; it is convenient to observe this changing the perspective of the sphere, manipulating the point Rot. When not draining the pixels of the screen, the plane has been illuminated partially. This causes that the image of the projection has illuminated in an incomplete way the sphere, giving him the appearance of a fishbowl made with the help of stamen; however, we can find branches of the spiral whose images are as close as it is wanted of the North Pole.

In spite of the above-mentioned, the fishbowl never empties then the branches of the image they never go by the North Pole of the sphere, although it conforms the branches of the spiral they move away from its center OR, the branches of the image give turns around the North Pole approaching more and more and to the pole. Another remarkable feature is that it conforms it diminishes the spacing of the branches of the spiral, the hank he/she closes more the spaces of our fishbowl.

This way, our exploration can be summarized in the following way:

6. The stereographic projection is a topologic map of the complete plane in a sphere; reason why the plane and the sphere are surfaces topologically equivalent,

It can already find in (Schwerdtfeger, 1979), justified from a very different focus, putting the emphasis in the geometric part of the result. Let us notice that we have given an argument of constructive type about the result. We have proposed a specific curve and a method to reach all the points of the complex plane (or if you want, we have given a family of curves that fill the plane). The images fill the representation that we have given of the sphere.
An algebraic formulation of the studied curve type is more convenient to make it expressing the curve in polar form. Do we have this way that if \( z^* \) does it represent the complex number that determines the point that belongs to the spiral of type archimedian, where \( k \) is a factor of proportionality that regulates the vicinity of the branches of the spiral and \( \theta \) is the angle of turn of the vector that unites the center of the spiral and \( z^* \), when the spiral is centered in the number complex fixed \( z_0 \), we can write this number \( z^* \) in the following way:

\[
z^* = r^* e^{i\theta^*} = k \theta e^{i\theta} + r_0 e^{i\theta_0} = (k \theta \cos \theta + r_0 \cos \theta_0) + i(k \theta \sin \theta + r_0 \sin \theta_0)
\]

In the way of \( z^* \) is it observed that if \( \theta \) tends to \( \infty \) then the module of \( z^* \), it also spreads to \( \infty \). The image corresponding to \( z^* \) tends to the North Pole as \( \theta \) tends to \( \infty \) because we observe that again:

\[
\lim_{\theta \to \infty} g(z^*) = \lim_{\theta \to \infty} \frac{4 \Re z^*}{\|z^*\|^2 + 4} + \lim_{\theta \to \infty} \frac{4 \Im z^*}{\|z^*\|^2 + 4} + \lim_{\theta \to \infty} \frac{2\|z^*\|^2}{\|z^*\|^2 + 4} = N(0,0,2)
\]

We can also formulate in a more classic way that we observed for the spiral: for each \( \epsilon \) positive exists a natural number \( N \), which the neighborhood of radio \( \epsilon \) centered in the North Pole of the surface of the sphere contains all the branches of the spiral starting from the
branch number $n$ with the condition that $n$ will be greater than $N$. Another question: with the starting point of the spiral fix, when does the area between the branches of the spiral cover to an hemisphere?

We have given arguments that justify the last traditional result notably in a computer environment, with nontraditional tools and resources to a mathematics’ class. Also, the first results also have a personality presented in an atmosphere of computer exploration, as we have seen it along this work.

**Conclusion**

An observation of didactic type is that the traditional group of results is preferably from an algebraic point of view; the graphic part leans on mainly in that know in the sphere the relating ones of those that one speaks (meridians, equator, parallel, hemispheres, poles, etc.). The emphasis in the algebraic part can be balanced by means of a bigger microworlds exploration whose mathematical properties are adjusted those of the material objects of which the first definitions, notions and relationships come. For it, we intend that the new curriculum glimpses a vast analysis of the possible groups that serve from domain to this transformation.

To accept the arguments that here it has been exhibited grace to the computer, they imply that we make a revision, in the first place, the role of the symbiosis between the computer representation of the objects and the adjacent notions. Later, to review the interaction class that is made with the students and the models; and certainly, the student’s conceptions, which are involved in the mathematical properties in the interaction with the microworld given through their perception.

**Bibliography**


